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Communication

Analytical Study of the Behavioral Trend of Klein-Gordon Equation in Different Potentials

Emmanuel Ifeanyi Ugwu[1, 2, *], Idu Hyacenth Kevin[2]

[1]Department of Physics, Nigerian Army University, Biu, Nigeria
[2]Department of Physics, Taraba State University, Jalingo, Nigeria

Abstract

In this work, we present the analysis of behavioral trend of Klein-Gordon Equation involving potential as regards when it comes to the study of particle, it has been observed that in every case of handling of KGE with potential of any type, it is made clear here that the equation has to first off all be transformed into a particular standard differential equation with a well-known solution which appears in form of implicitly defined transcendental equation. The equation on the other hand is to be solved analytically since the exact solution is not easily attainable without the use of mathematical tool especially when it comes to the consideration of the energy eigenvalue and the corresponding wave function because the solution is also always accompanied with a normalization constant often coupled with a condition that requires an arbitrarily chosen quantum number that come up when (l=0) and so on. In general, the analysis reveals the fact that the of trend of KGE involving potential gives a good understanding in the study of inter-molecular structure, diatomic crystals, and such case that involves inter-atomic interaction which is gives very nice idea in the study of bound state in atom.

Keywords

Analysis, Klein-Gordon Equation, Differential Equation, Potential, Transformation, Particle, Eigenvalue, Wave-Function, Solution, Normalization

1. Introduction

Many authors have used one form potential out of different known potentials to solve Klein-Gordon or Dirac relativistic equation in order to study free particle behavior or to ascertain its energy Eigen value [1-9] the case of KGE, it describe so well spin-zero particle if potential is applied because it enable the reduction of complex KGE to solvable equation state such that the exact solution or the analytical solution can be obtained [10-14]. With this, it still requires the use of good mathematical methods such as the variational method, functional analysis, supers metric approach, Nikiforov-Uvarov (NU, the asymptotic iteration method and so on In recent period, work has been carried out on to study bound state of KGE for a number of special potentials [1, 15] even in the case of equal vector and scalar potential [16], because it reduces KGE to a Schrodinger type of equation which could in turn be transformed into hypergeometric differential equation that has a known solution using [17-20] and this is more reason why KGE is receiving attention considerably in the literature when it comes to use of potentials [21, 22]. In fact it has been shown that exact solution are possible with some

*Corresponding author: emmanuel.ifeanyi@naub.edu.ng (Emmanuel Ifeanyi Ugwu), ugwuei2@gmail.com (Emmanuel Ifeanyi Ugwu)

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certain central potentials [23-26] which has help in investigation of bound states of the KGE for that particular potential which on the other hand has invariably led to derivation of the exact expression of the energy eigenvalues and the corresponding normalized eigenfunctionns in terms of some special polynomials and hypergeometrical function [16, 27-29]. In this work we intend to analyze the trend of Klein- Gordon Equation when different potential is applied into it to study particle as the equation is complicated to be used in isolation in the study of particle.

2. Theoretical Procedure

Here we first present the mathematical incursion by first of all considering spherical symmetric coulomb potential in Klein-Gordon Equation in order to write it in the form of stationary Klein-Gordon equation as given below.

$$[(\varepsilon - V(r) - m_0^2 c^o]\psi(\mathrm{r}) = 0 \tag{1}$$

$$-\hbar c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2 \sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right)\right]\psi(r)$$

$$= \left[\left(\varepsilon - V(r)\right)^2 - m_o^2 c^4\right]\psi(r) \tag{2}$$

Separating the equation in terms of the wave function, we have

$$\psi(r) = u(r)Y(\theta,\phi) \tag{3}$$

That leads to

$$(\hbar c)^2 \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial u}{\partial r} + \{(\varepsilon - V(r))^2 - m_o^2 c^4 - \frac{\lambda}{r^2}\}u = 0 \tag{4}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta}\,\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y = 0 \tag{5}$$

The solution to the last equation is spherical harmonic

$$Y_{lm} = l(l+1),\ \ \mathrm{l=0,\ 1,\ 2,\ and}\ \ Y_{lm} = l(l+1), \tag{6}$$

This leads to radial differential

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right)\right]u(r) = \frac{(\varepsilon - V)^2 - m_0^2 c^4}{\hbar c} \tag{7}$$

And then with $u(r) = \frac{R(r)}{r}$, equation 6 is transformed into

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right]R(r) = 0 \tag{8}$$

$$\text{Where}\ \ k^2 = \frac{(\varepsilon - V)^2 - m_o^2 c^2}{(\hbar c)^2} \tag{9}$$

Equation 7 is a confluent of hypergeometric differential equation.

2.1. Klein-Gordon Equation for Exponential Potential

Now for Klein-Gordon equation for exponential potential, we consider the potential in the form of $V(r) = -Z\alpha e^{-r/a}$ for which α represent the Summerfield fine structure constant that characterizes the range of the potential, and then with restriction to state and (l=0) the KGE becomes

$$\left(\frac{d^2}{dr^2} k^2\right)R(r) = 0 \tag{10}$$

Where k^2 has been defined by equation 8

With $R(r) = e^{r/2a} w(t)$ and by substituting $t = 2iz\alpha \frac{a}{\hbar c} e^{-r/a}$, equation 9 is transformed to

$$\frac{d^2 w}{dt^2} + \{-\frac{1}{4} - \frac{i\varepsilon a}{\hbar ct} + \frac{1/4 - p^2 a^2}{t^2}\}w = 0 \tag{11}$$

$$\text{Where}\ \ p^2 = \frac{m_o^2 c^4 - c}{\hbar c^2}$$

Equation (10) in its form stands for Wittaher differential equation that has a regular solution as given below with N as its normalization factor

$$w(t) = Nw\lambda_\mu(t) = Ne^{-1/2} t^{1/2+\mu} \tag{12}$$

$${}_I F_I\left(\frac{1}{2} + \mu - \lambda, 1 + 2\mu; t\right) \tag{13}$$

for $r \to \infty$ when t=0

where $\lambda = \frac{i\varepsilon a}{\hbar c}$, and $\mu = pa$

Thus leading to

$$R(r) = Ne^{r/2a} w_{I,\mu}(2iz\alpha \frac{a}{\hbar c} e^{-r/a} \tag{14}$$

To deduce the eigenvalue equation for the solution of the energy eigenvalue, R(r) has to vanish at r=0 in order to guarantee the normalization of the radial wave function and this will result to the eigenvalue equation

$$ {}_I F_I\left(\frac{1}{2}+\mu-\lambda, 1=2\mu; 2iz\alpha\frac{a}{\hbar c}\right)=0 \qquad (15)$$

This equation is implicit in nature and as such for the determination of the energy eigenvalue ε, the solution can only be handled by using numerical approach.

2.2. KGE for Scalar Potential

In this case we consider $1/r$ potential with scalar interaction of the form $w(\ell)=z\frac{\alpha}{r}$ which is coupled to the square of mass directly with a long range $1/r$ interaction to yields a transformed KGE with an arbitrary scalar interaction of the form

$$\left[\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{\varepsilon^2}{(\hbar c)^2}+\frac{(m_o c^2)^2}{(\hbar c)^2}-\frac{u^2(r)}{(\hbar c)^2}\right]R(r)=0 \qquad (16)$$

Further simplification is carried out by making the following substitution

$r'=r\frac{m_o c^2}{\hbar c}$, and this transforms the equation into the following form

$$\left[\frac{d^2}{dr'^2}+\frac{l(l+1)}{r'^2}+\frac{\varepsilon^2}{(m_o c^2)^2}-1-\frac{u^2(r')}{(m_o c^2)^2}\right]R(r')=0 \qquad (17)$$

Now we simplify the interaction by the making use of by making use of the followings

$$u^2(r')=\mathrm{w(r')}=\frac{z\alpha}{r'} \quad \text{and} \quad b^2=1-\frac{\varepsilon^2}{(m_o c^2)^2}$$

Equation 16 becomes

$$\left[\frac{d^2}{dr'^2}-\frac{l(l+1)}{r'^2}-b^2+\frac{z\alpha}{r'}\right]R(\mathrm{r}') \qquad (18)$$

Then going further with $\ell=2br'$, we get

$$\left[\frac{d^2}{d\ell^2}-\frac{l(l+1)}{\ell}-\frac{1}{4}+\frac{\ell}{c}\right]R(\ell)=0 \qquad (19)$$

Where $c=\frac{z\alpha}{2b}$

At $\ell\to\infty$ and $\ell\to 0$, equation 18 is decomposed into two;

$$\left[\frac{d^2}{d\ell^2}-\frac{1}{4}\right]R(\ell) \qquad (20)$$

Having a solution of the form $R(\ell)\propto e^{-\frac{\ell}{2}}$

And the other one

$$\left[\frac{d^2}{d\ell^2}-\frac{l(l+1)}{\ell^2}\right]R(\ell)=0, \text{ with solution } R(\ell)\propto\ell^{l+1} \qquad (21)$$

Thus the general solution of the system of the equation becomes

$$R(\ell)=N\ell^{l+1}F(\ell)e^{-\ell/2} \qquad (22)$$

Where $F(\ell)$ is yet to be determined.

Putting equation 21 into equation 18, leads to

$$\ell\frac{d^2}{d\ell^2}+[2l+2-\ell]\frac{\mathrm{dF}(\ell)}{d\ell}[c-(l+1)]F(\ell)=0 \qquad (23)$$

This equation is known as equation Kummer's differential equation with a known solution given

$$\text{as } F(\ell)={}_I\mathrm{F}_I(l+1-c, 2l+2, \ell) \qquad (24)$$

with the energy eigenvalue given as

$$\varepsilon=\mp\{1-\frac{(z\alpha)^2}{4n^2}\}m_o c^2 \qquad (25)$$

This evidentially shows that the orbital quantum number l is not involved while the solution of the Laguerre polynomial yields the radial wave function

2.3. KGE for Square- Well Potential

This involves coupling minimally potential V to KGE in order facilitate the transformation of the equation into solvable state. The potential is given as

$$V=\{{}^{V_o\, for\cdots r\le R}_{0\, for\ldots r>R} \qquad (26)$$

Whereby we set $k_1=\frac{1}{\hbar c}\sqrt{\varepsilon^2-(m_o c^2)^2}$ for $r\le R$ (27)

and

$$k_o = \frac{1}{\hbar c}\sqrt{\varepsilon^2 - (m_o c^2)^2} \text{ for } r > R \tag{28}$$

KGE is transformed using equation 27 into

$$\left[\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - l(l+1) + (kr)^2\right]u(r) = 0 \tag{29}$$

Setting $\ell = kr$, we differentiate to obtain

$$\frac{d}{dr} = k\frac{d}{d\ell} \tag{30}$$

$$\text{And } \frac{d^2}{d\ell^2} = k^2\frac{d^2}{d\ell^2} \tag{31}$$

This in turn led to

$$\left[2\ell\frac{d}{d\ell} + \ell^2\frac{d^2}{d\ell^2} - l(l+1) + \ell\right]u(\ell) = 0 \tag{32}$$

Represents Bessel equation whose solution for $r \le R$ is

$$u(\ell) = N_{ji}(k_j r) \tag{33}$$

With exclusion of the Neumann function due to its regular value at $\ell = 0$ for V=0, the energy eigenvalue is deduced numerically from equation 31 by considering the equality of logarithmic derivatives of the solution from recurrence solution obtained from Bessel equation at r=R, using iteration method [19-22] with N being the normalization factor which is symmetric for positive and negative energy eigenvalue where

$$k = \frac{1}{\hbar c}\sqrt{\varepsilon^2 - (m_o c^2)^2} \tag{33}$$

3. Analysis

This analysis indicates that direct application of KGE in the study of particle could not yield exact solution tenable for appropriate description of particle especially, spin-0 particle as expected.

Rather it was more emphatic that the equation needs first of all to be transformed into one form of differential equation or the other that has a known solution or that can be handled more easily using one form of mathematical method or the other. [9-12]

In the first case here that involves spherical coulomb potential, the KGE is needed to be transformed into hypergeometric differential equation as shown equation 8 and for the one that involves exponential potential, a transformation has to be into Wittaher's differential equation as shown in equation 9, while the last two as presented here are needed to be transformed into Kummer's and Bessel's differential equation respectively to easy out the solution since some of them have one form of known solution in form of polynomial whose coefficient forms a set of elementary symmetric polynomials, and whose solution is based on one mathematical tool an approach that has simplicity and flexibility in leading to approximation and analysis of the eigenvalue and eigenfunction as exact solution is not attainable with KGE when it comes the behavior of the particle in any type of potential

4. Conclusion

From this analysis, it was clearly showcased that KGE involving potential meant to study particle requires a transformation before the deduction of an expression for the energy eigenvalue and the corresponding wave function which of course has to be achieved from the transformed equation whose solution is often implicitly defined transcendentally in nature and as such requires mathematical tool that can handle such a case analytically without restriction as in the literature [13-14]. In general, this reveals the fact that the analysis of trend of KGE involving potential gives good understanding in the study of inter-molecular structure, diatomic crystals and such case that involves inter-atomic interaction [30-31] and also very good in the study of bound state in atom.

Conflicts of Interest

The authors declare no conflicts of interest.

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